nozzle. Gimballing a portion of the plug appears to be relatively inefficient, particularly since that portion of the plug which is gimballed can be removed with no decrease in performance. The most promising methods of vectoring appear to be throttling or gimballing the individual nozzles with the shortest plug length consistent with satisfactory unvectored performance.

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Falkner-Skan Equation for Wakes

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THE existence of physically acceptable solutions of the Falkner-Skan equation

$$f''' + ff'' + \beta(1 - f'^2) = 0 \tag{1}$$

subject to the boundary conditions

$$f(0) = f''(0) = 0 \qquad f'(\infty) = 1 \tag{2}$$

in the range $-\frac{1}{2} \leqslant \beta < 0$ was pointed out by the present author some ten years ago.[1] These solutions have some relevance to the flow in the laminar wake behind a flat-based body, as has been noted by Kennedy,[2] who has also carried out further and more accurate integrations of the equation. From a study of the solutions obtained, he conjectures that as $\beta \to 0-$

$$f' \to f_0' \tag{3}$$

apart from a shift of origin, where f_0 is the solution of (1), with $\beta = 0$, and satisfying

$$f_0'(-\infty) = 0 \qquad f_0'(\infty) = 1 \tag{4}$$

This function has been computed by Chapman.[3]

Further points of interest from the numerical studies[1, 2] are that, if $-0.1988 < \beta < 0, f'(0) < 0$, which means that the flow direction in the wake is reversed, and that as $\beta \to 0-$, $f'(0) \to 0-$ having a vertical tangent there. An estimate of the dependence of $f'(0)$ on β near $\beta = 0$ has been given earlier,[1] and in this note an improved version is obtained.

We take (3) as our starting point and show that a consistent solution can be derived from it when β is small. Choose the origin of the independent variable η in (1) so that as $\eta \to -\infty$

$$f_0 = -q + qe^{q\eta} - \frac{q}{4}\, e^{+2q\eta}\, \frac{5q}{72}\, e^{3q\eta} \ldots \tag{5}$$

where $q = 0.876\ldots$. In consequence, the boundary conditions in (2) are that f and f'' vanish together at some value η^* of η, to be determined. From the numerical integrations, it may be expected that $\eta^* \to -\infty$ as $\beta \to 0-$. The implication of (3) is that, when $|\beta|$ is small and $\eta = O(1)$, we can write

$$f = f_0 + af_1 \tag{6}$$

where a is a small number. Substituting in (1),

$$f_1''' + f_0 f_1'' + f_1 f_0'' = -(\beta/a)(1 - f'^2) - af_1 f_1'' \tag{7}$$

and we now make the further assumption that $|\beta| \ll a$. The right-hand side of (7) is then negligible in the limit $a \to 0$, and the general solution for f_1 is

$$f_1 = Af_0' + B(\eta f_0' + f_0) + C(\eta f_0' + f_0) \times \int_\eta^\infty \frac{f_0' f_0'' d\eta}{(2f_0'^2 + f_0''')^2} - Cf_0' \int_\eta^\infty \frac{f_0''(\eta f_0' + f_0)d\eta}{(2f_0'^2 + f_0''')^2} \tag{8}$$

where A, B, C are constants. Since $f_1'(\infty) = 0$, $B = 0$, and hence, on substituting the known values of f_0 and integrating, we find that as $\eta \to -\infty$

$$f_1 = (\eta + b)C/q^2 + O(\eta e^{\eta q}) \tag{9}$$

where $b = 1.765/q = 2.014$. Writing $\alpha = -aC/q^2$, it follows that

$$f + \alpha\eta \to -q - \alpha b + O(\alpha^2, \beta) \tag{10}$$

as $\eta \to -\infty$. It is noted that if $\alpha < 0, f > 0$ for sufficiently large and negative η. In turn, this implies that f'' ultimately increases exponentially, so that the expansion (5) is not uniformly valid for all η. However, we are only interested in values of $\eta > \eta^*$, at which point f vanishes, but we need to insure that here f'' also vanishes. When $\eta \ll -1, f'^2 \ll 1$, and the governing equation takes on the form

$$f''' + ff'' = -\beta \tag{11}$$

We can no longer neglect β, because f'' is exponentially small when $\eta^* < \eta \ll -1$, but this means that to a first approximation we can replace f in (11) by (10), so that (11) becomes

$$f''' - (q + \alpha b + \alpha\eta)f'' = -\beta \tag{12}$$

with solution

$$f'' = D \exp\left[\frac{1}{2\alpha}(\alpha\eta + \alpha b + q)^2\right] - \beta \times \exp\left[-\frac{1}{2\alpha}(\alpha\eta + \alpha b + q)^2\right] \times \int_0^\eta \exp\left[\frac{1}{2\alpha}(\alpha\eta' + q + \alpha b)^2\right] d\eta' \tag{13}$$

Now when $\eta = O(1)$ but large,

$$f''' \approx q^3 e^{q\eta} \tag{14}$$

from (5), whence

$$D \doteqdot q^3 \exp\left[-\frac{1}{2\alpha}(q + \alpha b)^2\right] \doteqdot q^3 \exp\left[-\frac{q^2}{2\alpha} - qb\right] \tag{15}$$

Furthermore, using (10), $f = 0$ when

$$\eta = \eta^* \doteqdot q/\alpha + b \tag{16}$$

and $f'' = 0$ at this point if

$$\beta = -D(2\alpha/\pi)^{1/2} \tag{17}$$

Having obtained an estimate for β in terms of α, the various assumptions made in the course of the argument can now be shown to be consistent. In particular, the neglect of β in (7) and the assumption that the perturbation of f from (10) is negligible even when $\eta = O(\alpha^{-1})$ is justified. We conclude that, when $f = f'' = 0$,

$$f' = -\alpha \tag{18}$$

whence, returning to the origin of η implied in (2), we have that, as $\alpha \to 0+$,

$$\beta \approx -q^3(2\alpha/\pi)^{1/2} \exp[-qb - q^2/2\alpha]$$

where

$$f'(0) \approx -\alpha \tag{19}$$

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Table 1

$-f'(0) \approx \alpha$	$-\beta$ Kennedy	$-\beta$ (19)
0.1637	10^{-2}	0.35×10^{-2}
0.1074	10^{-3}	0.85×10^{-3}
0.0713	10^{-4}	1.15×10^{-4}
0.0513	10^{-5}	1.21×10^{-5}
0.0396	10^{-6}	1.17×10^{-6}

In the Table 1, a comparison is made between Kennedy's numerical results and those predicted by (19).

It may be noted that, for $|\beta| \leqslant 10^{-3}$, the errors in (19) correspond to errors $O(\alpha^2)$ in $f'(0)$ and may well arise from the neglect of α^2 in (7).

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Hydromagnetic Flow between Two Rotating Cylinders

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RAMAMOORTHY[1] has considered the hydromagnetic flow between two concentric rotating cylinders subject to a radial magnetic field. He has used the approximate equation deduced by Rossow,[2] neglecting the induced magnetic field. The simultaneous solution of the Navier-Stokes equations and Maxwell's equations for any hydromagnetic problem presents a formidable task. The induced magnetic field is generally ignored in order to uncouple the two sets of equations. The object of this note is to demonstrate that, in the problem of axisymmetric rotational flow in an annular channel, the two sets of equations do not get coupled even when the problem is generalized so that 1) the induced magnetic field is not ignored, 2) an axial pressure gradient is present, 3) the outer cylinder has a translational velocity in the axial direction besides the uniform rotational velocity, and 4) a uniform suction velocity is imposed at the wall of outer cylinder and a uniform injection velocity on the wall of inner cylinder.

In this generalized problem, it is demonstrated that the principle of independence of axial and rotational field holds. Generalized Couette-type flow caused by the relative motion of the walls of the channel, together with an axial pressure gradient with suction at the walls in the absence of magnetic field, has been considered by various authors.[3–6] The case of flow due to the axial motion of the outer cylinder with suction at the walls and an axial pressure gradient has been studied by Jain and Mehta.[7] These all become special cases of the problem considered here.

We consider steady-state laminar flow of an incompressible, viscous, electrically conducting fluid through an annulus with a and b as its inner and outer radii. Since the annulus is of infinite length in both directions (z axis) with no entry region of the fluid into the annulus, suction rate at the wall of the outer cylinder must be equal to the injection rate at the inner cylinder, i.e., $aV_a = bV_b$, where V_a and V_b are velocities of fluid injection and withdrawal. Cylindrical polar coordinates (r,ϕ,z) are used having the axis of channel as the z axis.

Since the problem is axisymmetric and the cylinders are infinite, the physical variables can be assumed to be independent of ϕ and z. They depend on the radial distance r only.

We assume that

$$\begin{aligned} \mathbf{V} &= [V_r(r),V_\phi(r),V_z(r)] \\ \mathbf{H} &= [H_r(r),H_\phi(r),H_z(r)] \end{aligned} \tag{1}$$

The divergence relation for $\mathbf{V}$ and $\mathbf{H}$ admit of a radial velocity and a radial magnetic field inversely proportional to r, that is,

$$V_r = bV_b/r \qquad H_r = A/r \tag{2}$$

where A is a constant.

Introducing an axial pressure gradient $-\partial p/\partial z = P =$ const, the magnetohydrodynamic equations in mks units take the following form:

Momentum

$$\frac{b^2V_b^2}{r^3} + \frac{V\phi^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\mu}{\rho}\left[\frac{H\phi^2}{r} + H_\phi\frac{dH_\phi}{dr} + H_z\frac{dH_z}{dr}\right] \tag{3}$$

$$\frac{d^2V_\phi}{dr^2} - (R-1)\frac{1}{r}\frac{dV_\phi}{dr} - (R+1)\frac{V_\phi}{r^2} + \frac{A\mu}{\rho\nu}\frac{H_\phi}{r^2} + \frac{A\mu}{\rho\nu}\frac{1}{r}\frac{dH_\phi}{dr} = 0 \tag{4}$$

$$\frac{d^2V_z}{dr^2} - (R-1)\frac{1}{r}\frac{dV_z}{dr} + \frac{A\mu}{\rho\nu}\frac{1}{r}\frac{dH_z}{dr} + \frac{P}{\rho\nu} = 0 \tag{5}$$

Ohm's Law

$$V_\phi = (\nu/A\varkappa)[(\varkappa R - 1)H_\phi - r(dH_\phi/dr)] \tag{6}$$

$$V_z = (\nu/A\varkappa)[\varkappa RH_z - r(dH_z/dr)] \tag{7}$$

where R is suction Reynolds number (bV_b/ν) and $\varkappa = \sigma\mu\nu$.

The boundary conditions are

$$\left.\begin{aligned} V_\phi &= b\omega_2 \\ H_\phi &= 0 \end{aligned}\right\} \text{ at } r = b \qquad \left.\begin{aligned} V_z &= U \\ H_z &= 0 \end{aligned}\right\} \text{ at } r = b \tag{8}$$
$$V_\phi = a\omega_1 \text{ at } r = a \qquad V_z = 0 \text{ at } r = a$$

Equations (4) and (6) are two simultaneous ordinary differential equations in V_ϕ and H_ϕ, and they determine the peripheral velocity and magnetic field. Equations (5) and (7) similarly determine the axial fields. The two pairs of equations can be solved independently of each other. Thus the axial and peripheral fields do not interact in the problem posed. The radial pressure distribution can be determined from Eq. (3) after V_ϕ and H_ϕ are obtained as solutions of Eqs. (4) and (6).

Eliminating V_ϕ between Eqs. (4) and (6) and V_z between Eqs. (5) and (7), we get

$$r^3\frac{d^3H_\phi}{dr^3} + (4 - R - \varkappa R)r^2\frac{d^2H_\phi}{dr^2} + [(R-1)(\varkappa R - 2) - (R+1) - M^2]r\frac{dH_\phi}{dr} + [(R+1)(\varkappa R - 1) - M^2]H_\phi = 0 \tag{9}$$

$$r^3\frac{d^3H_z}{dr^3} + (3 - R - \varkappa R)r^2\frac{d^2H_z}{dr^2} + [(R-1)(\varkappa R - 1) - M^2]r\frac{dH_z}{dr} - \frac{A\varkappa}{\rho\nu^2}r^2P = 0 \tag{10}$$

where $M^2 = A^2\mu^2\sigma/\rho\nu$. The sets of three boundary condi-

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